Prime avoidance

Thm (Prime avoidance) Let $J, I_1, \ldots, I_n$ be ideals in R. Suppose $J \subseteq \bigcup_j I_j$. If R contains an infinite field or if at most two of the I_j are not prime, then J is contained in some I_j.

(If R is graded, J is gen. by homogeneous elements of degree >0, and all the I_j are prime, then it's enough to assume the homogeneous elts of J are in $\bigcup I_j$.)

Note: If I is not contained in any of a finite number of primes, the theorem says there is some $x \in I$ that "avoids" all the primes, hence "prime avoidance".

Pf of theorem: First suppose R contains an infinite field. Then J is a k-vector space, so if it's contained in the union of finitely many subspaces it must be contained in one of them.

Now, if at most two I_j are not prime, we do induction on n. If $n=1$, it's obvious.

If J is in any smaller union of the I_j, we're done by induction. Thus, assume it's in no smaller union, so we can find, for each i, $x_i \in J$ s.t. $x_i \in I_i$ but

$x_i \notin I_j,\ j \neq i.$

If $n=2$, then $x_1 + x_2$ is not in I_1 or I_2, which is a contradiction.

If $n>2$, assume I_1 is prime. Then $x_1 + \underbrace{x_2 x_3 \cdots x_n}_{\text{not in } I_1}$ is not in any I_j, again a contradiction.

This along with the previous theorem implies the following:

Cor: R Noetherian, $M \neq 0$ a finitely generated R-module. Let $I \subsetneq R$ be an ideal. Either I contains a nonzerodivisor on M or I annihilates an element of M.

Pf: If $I \not\subset \bigcup_{P \in \operatorname{Ass} M} P$, then I contains a nonzerodivisor on M. Otherwise $I \subset P = \operatorname{Ann}(x)$, some $x \in M$. $\square$

Now we work on the proof of the theorem in the previous section. First we show the following:

Prop: R a ring, $M \neq 0$ an R-module. If $I \subset R$ is max'l among ideals of R that are annihilators of elements of M, then I is prime. In particular, if R is Noetherian, $\operatorname{Ass} M \neq \emptyset$.

Pf: Let $a, b \in R$ s.t. $ab \in I$. Let $I = \operatorname{ann}(x)$.

Suppose $b \notin I$. Then $bx \neq 0$, but $abx = 0$.

$\Rightarrow (I, a) \subseteq \operatorname{ann}(bx)$. By maximality, $a \in I$, so I is prime. $\square$

Note that this immediately proves part b.) of the theorem from the previous section, which says

$$\text{If } R \text{ is Noetherian, } \bigcup_{P \in \operatorname{Ass} M} P = \{\text{zerodivisors on } M\} \cup \{0\}$$

This also makes it easier to check whether or not elements of a module M are 0. Recall that we showed previously that $x \in M = 0 \iff x \mapsto 0$ in M_m for every max'l (or just prime) ideal $m \subseteq R$.

But if R is Noetherian, we can restrict our attention to associated primes. More precisely:

Cor: Let M be an R-module, R Noetherian.
If $x \in M$, then $x = 0 \iff$ the image of x is 0 in each M_P for each of the maximal associated primes P of M.

Pf: $\Rightarrow$ is already done.

Suppose $x \neq 0$. Then since R is Noetherian $\exists$ a prime $P \in \operatorname{Ass} M$ that is max'l among annihilators of elts containing $\operatorname{Ann} x$. Thus, $x/1 \neq 0$ in M_P. $\square$

We can now observe how taking associated primes acts in short exact sequences.

Lemma: Let R be Noetherian. If

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of R-modules, then

$$\operatorname{Ass} M' \subseteq \operatorname{Ass} M \subseteq (\operatorname{Ass} M') \cup (\operatorname{Ass} M'').$$

Pf: The first containment is clear.

For the second, let $P \in \operatorname{Ass} M \setminus \operatorname{Ass} M'$. If $P = \operatorname{ann} x$, then $Rx \cong R/P$.

For $0 \neq \bar{y} \in R/P$, $a\bar{y} = 0 \iff a \in P$, since P is prime. Thus, every nonzero element of Rx also has annihilator P.

$\implies Rx \cap M' = 0 \Rightarrow Rx$ is isomorphic to its image in $M'' \Rightarrow P \in \operatorname{Ass} M''$. $\square$

To finish the proofs of part a.) of the theorem in the previous section, we now just need the following:

Prop: If R is Noetherian and M is a finitely generated R-module, then M has filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$$

w/ each $M_{i+1}/M_i \cong R/P_i$ for some prime ideal P_i.

Pf: Since R is Noetherian, if $M \neq 0$, $\operatorname{Ass} M$ is nonempty. Let $P_1 \in \operatorname{Ass} M$, so there's a submodule $M_1 \cong R/P_1$. Repeating this w/ M/M_1, we get M_2. This process terminates since M is Noetherian. $\square$

We also need the following lemma, which essentially says that taking associated primes commutes w/ localization:

lemma: If R is Noetherian and M an R-module, we have

$$\operatorname{Ass}(U^{-1}M) = \{ QU^{-1}R \mid Q \in \operatorname{Ass} M \text{ and } Q \cap U = \emptyset \}$$

Pf: Suppose $Q \in \operatorname{Ass} M$ and $Q \cap U = \emptyset$. Then $Q = \operatorname{Ann}(x)$ for some $x \in M$.

Suppose $\frac{a}{u} x = 0$ in $U^{-1}M$. Then $(va)x = 0$ for $v \in U$.
$\Rightarrow va \in Q$. But $v \notin Q$, so $a \in Q$.

Thus, $QU^{-1}R \in \mathrm{Ass}(U^{-1}M)$.

Conversely, suppose $P \in \mathrm{Ass}(U^{-1}M)$. Then $P = Q(U^{-1}R)$ for some $Q \subseteq R$ prime not meeting U.

Say $P = \mathrm{Ann}_{U^{-1}R}\left(\frac{x}{u}\right) = \mathrm{Ann}_{U^{-1}R}(x)$.

Then for each $q \in Q$, $\frac{qx}{1} = 0$, so there is $u_q \in U$ s.t. $u_q q x = 0$.

Since R is Noetherian, Q is finitely generated, say $Q = (q_1, \ldots, q_n)$. Set $u = u_{q_1} u_{q_2} \cdots u_{q_n}$.

Then $\forall\ q \in Q$, $q(ux) = 0$. Thus, $Q \subseteq \mathrm{Ann}_R(ux)$.

So we have:

$$Q \subseteq \mathrm{Ann}_R(ux) \subseteq \mathrm{Ann}_{U^{-1}R}(ux) \cap R \subseteq \mathrm{Ann}_{U^{-1}R}(x) \cap R = P \cap R = Q$$

Thus, $Q = \mathrm{Ann}_R(ux)$, so $Q \in \mathrm{Ass}_R M$. $\square$

Now we conclude the proof of the theorem from last section. Recall part a.) of the theorem, again under the assumption that R is Noetherian and $M \neq 0$ is f.g.:

a.) Ass M is finite and nonempty, each containing ann(M). It includes all primes minimal among those containing ann M.

Pf: For the finiteness statement, we give a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_n = M, \quad \text{where} \quad M_{i+1}/M_i \cong R/P_i, \; P_i \text{ prime.}$$

We prove by induction on n. For $n=1$, $M \cong R/P_i$

So $\forall \underset{0\neq}{x} \in M$, $ax = 0 \iff a \in P_i$, so $\mathrm{Ass}_R M = \{P_i\}$.

For $n>1$, $\mathrm{Ass}\, M \subseteq \underset{\substack{\uparrow \\ \text{finite by} \\ \text{induction}}}{\mathrm{Ass}\, M_{n-1}} \cup \underset{\substack{\shortparallel \\ \{P_{n-1}\}}}{\mathrm{Ass}\left(R/P_{n-1}\right)} \Rightarrow \mathrm{Ass}\, M$ is finite.

Now we just need to show that if P is a prime ideal minimal over Ann M, $P \in \mathrm{Ass}\, M$.

By the lemma, $\mathrm{Ass}_{R_P} M_P = \{QR_P \mid Q \in \mathrm{Ass}\, M \text{ and } Q \subseteq P\}$

But PR_P is the only prime in R_P containing Ann M and since $\mathrm{Ass}_{R_P} M_P \neq \emptyset$, $P \in \mathrm{Ass}\, M$. $\square$